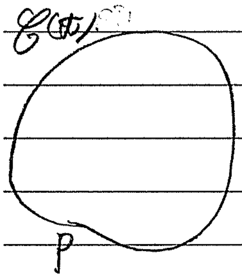
Lecture 1. 曲率

曲線の分類

単純閉曲線



simple closed curve

始点と終点が一致し、自己交叉点をもたない

$C(\alpha) = C(\beta)$

$I \ni t$

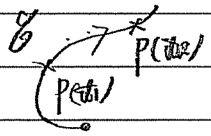
Jordan 曲線

Jordanの閉曲線定理

閉曲線でないとき、単純曲線　有向弧 oriented arc

$I \ni t_1, t_2$

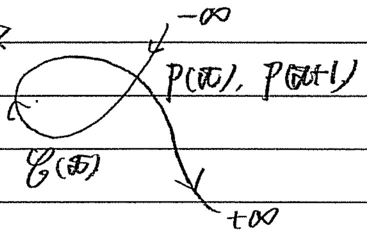
全単射(連続)



$t_1 < t_2 \Leftrightarrow P(t_1) < P(t_2)$

自己交叉があるとき

$t \in \mathbb{R}$



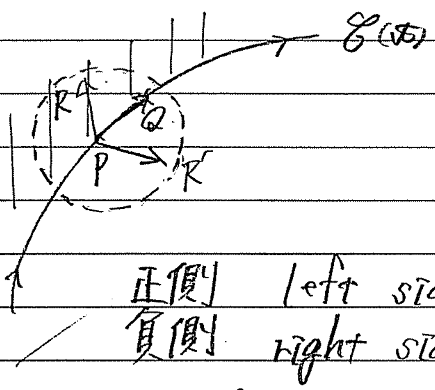
$C_1\ (-\infty, t]$

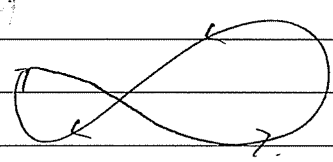
$C_2\ (t, t+1]$

$C_3\ (t+1, +\infty)$

C_1, C_2, C_3 : 有向単純弧

曲線の正側、負側





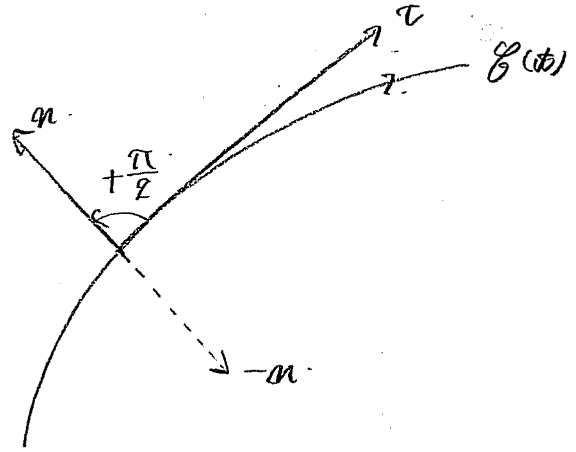
正側 left side

負側 right side

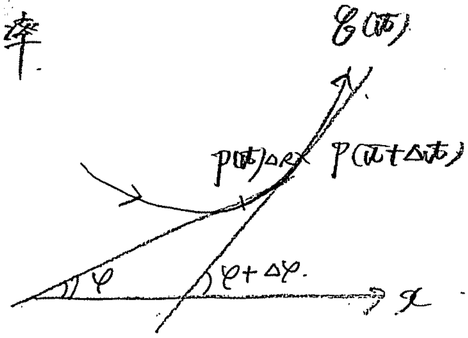
$PQ \times PR$ が上向き

自己交叉点があると

正側と負側が入れかわる



曲率



弧長を Δs
偏角の変化を $\Delta\varphi$ としたとき

$$\frac{\Delta\varphi}{\Delta s} \xrightarrow[\Delta s \to 0]{\Delta t \to 0} \frac{d\varphi}{ds}$$

これを $\mathscr{C}$ の P における曲率 curvature κ.

$\mathscr{C}(t): \begin{cases} x = x(t) \\ y = y(t) \end{cases}$ について $\tan\varphi = \dfrac{\dot{y}}{\dot{x}}$ より $\arctan\dfrac{\dot{y}}{\dot{x}} = \varphi$.

$$\frac{d\varphi}{dt} = \frac{1}{1+\left(\frac{\dot{y}}{\dot{x}}\right)^2}\cdot\frac{\ddot{y}\dot{x}-\dot{y}\ddot{x}}{\dot{x}^2} = \frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{\dot{x}^2+\dot{y}^2} = \frac{\begin{vmatrix} \dot{x} & \ddot{x} \\ \dot{y} & \ddot{y} \end{vmatrix}}{|\tau|^2}$$

$$\frac{ds}{dt} = \sqrt{\dot{x}^2+\dot{y}^2} \text{ より } \frac{dt}{ds} = \frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}}$$

よって $\kappa = \dfrac{d\varphi}{ds} = \dfrac{d\varphi}{dt}\cdot\dfrac{dt}{ds}$

$$= \frac{\begin{vmatrix} \dot{x} & \ddot{x} \\ \dot{y} & \ddot{y} \end{vmatrix}}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}} = \frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{|\tau|^3}$$

単位円 U について、$\kappa = 1$.
半径 a の円　$\kappa = \dfrac{1}{a}$

曲率半径
$\mathscr{C}(t)$ の点における曲率が κ のとき
その点における曲率半径 $\rho = \dfrac{1}{|\kappa|}$

Task 1.1

(2) Astroid　$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$.　$\mathscr{C}$

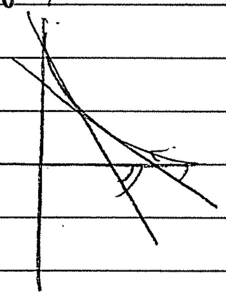
$x^{\frac{1}{3}} = \cos\theta$, $y^{\frac{1}{3}} = \sin\theta$ として

$\mathscr{C}$ を p-表示　$\begin{cases} x = \cos^3\theta \\ y = \sin^3\theta \end{cases}$

$\dot{x} = -3\cos^2\theta\sin\theta$.　$\dot{y} = 3\sin^2\theta\cos\theta$

$\ddot{x} = -3\cos\theta(\cos^2\theta - 2\sin^2\theta)$,　$\ddot{y} = 3\sin\theta(2\cos^2\theta - \sin^2\theta)$

∴ $\begin{vmatrix} \dot{x} & \ddot{x} \\ \dot{y} & \ddot{y} \end{vmatrix} = \dot{x}\ddot{y} - \ddot{x}\dot{y} = \cdots = -9\sin^2\theta\cos^2\theta\ (<0)$



Astroidの曲率中心の軌跡もまたAstroid

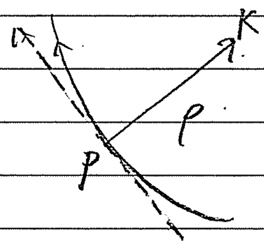
分母　$\dot{x}^2 + \dot{y}^2 = 9\sin^2\theta\cos^2\theta$ より

$(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}} = 27\sin^3\theta\cos^3\theta$

∴ $K = \dfrac{-1}{3\sin\theta\cos\theta}$　よって　$\rho = |3\sin\theta\cos\theta|$

曲率中心 K の軌跡 $\mathcal{L}(K)$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3\sin\theta\cos^2\theta \\ 3\sin^2\theta\cos\theta \end{pmatrix} = 3\sin\theta\cos\theta \overset{u_\theta}{\begin{pmatrix} -\cos\theta \\ \sin\theta \end{pmatrix}}$$



$\theta \neq 0, \frac{\pi}{2}$ ならば 接vector　$t = \begin{pmatrix} -\cos\theta \\ \sin\theta \end{pmatrix}$

K(ξ, η) として　$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cos^3\theta \\ \sin^3\theta \end{pmatrix} + 3\sin\theta\cos\theta \begin{pmatrix} -\cos\theta \\ \sin\theta \end{pmatrix}$

$\varepsilon : \begin{cases} \xi = \cos^3\theta + 3\cos^2\theta\sin\theta \\ \eta = 3\cos\theta\sin^2\theta + \sin^3\theta \end{cases}$

曲率円は

$(x - \xi)^2 + (y - \eta)^2 = 9\sin^2\theta\cos^2\theta$

ε を追跡すると

$\xi + \eta = (\cos\theta + \sin\theta)^3$.　$\xi - \eta = (\cos\theta - \sin\theta)^3$

ξ を $\frac{\pi}{4}$ 回転させる　　　　function view トロコイド

$$R_{\frac{\pi}{4}}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\xi-\eta\\ \xi+\eta\end{pmatrix}$$

$\frac{1}{\sqrt{2}}(\xi-\eta) = \bar{x}$，$\frac{1}{\sqrt{2}}(\xi+\eta) = \bar{y}$ とすれば

$\xi-\eta = \sqrt{2}\,\bar{x}$，$\xi+\eta = \sqrt{2}\,\bar{y}$

$$\therefore (\xi+\eta)^{\frac{2}{3}} + (\xi-\eta)^{\frac{2}{3}} = (\cos\theta+\sin\theta)^2 + (\cos\theta-\sin\theta)^2 = 2$$

$$(2^{\frac{1}{2}}\bar{x})^{\frac{2}{3}} + (2^{\frac{1}{2}}\bar{y})^{\frac{2}{3}} = 2 \qquad \therefore \bar{x}^{\frac{2}{3}} + \bar{y}^{\frac{2}{3}} = 2^{\frac{2}{3}}$$

(3) γ_3: $\begin{cases} x = a(t-\sin t)\\ y = a(1-\cos t)\end{cases}$ Cycloid

$a=1$.　$\dot{x} = 1-\cos t$，$\dot{y} = \sin t$
　　　$\ddot{x} = \sin t$，$\ddot{y} = \cos t$

$$\dot{x}^2+\dot{y}^2 = (1-\cos t)^2 + \sin^2 t = 2(1-\cos t)$$

$$\dot{x}\ddot{y} - \ddot{x}\dot{y} = \cos t(1-\cos t) - \sin^2 t = \cos t - 1$$

$$\therefore K = \frac{-1(1-\cos t)}{\{2(1-\cos t)\}^{\frac{3}{2}}} = \frac{-1}{2^{\frac{3}{2}}\sqrt{1-\cos t}}$$

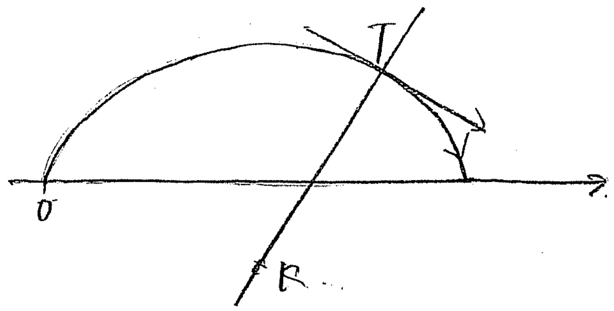
$1-\cos t = 2\sin^2\frac{t}{2}$ より $K = \dfrac{-1}{2^2\left|\sin\frac{t}{2}\right|}$

よって $\rho = 2^2\left|\sin\frac{t}{2}\right|$.

接vector

$$\tau = \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}1-\cos t\\ \sin t\end{pmatrix}$$

$$n = \begin{pmatrix}\sin t\\ \cos t - 1\end{pmatrix}$$



$$|n|^2 = 4\sin^2\frac{t}{2} \qquad \therefore |n| = 2\left|\sin\frac{t}{2}\right|$$

$$\overrightarrow{OK} = \overrightarrow{OT} + \overrightarrow{TK} = \begin{pmatrix}t-\sin t\\ 1-\cos t\end{pmatrix} + \overbrace{4\left|\sin\frac{t}{2}\right|}^{\rho}\,\overbrace{\frac{1}{2\left|\sin\frac{t}{2}\right|}\begin{pmatrix}\sin t\\ \cos t-1\end{pmatrix}}^{\frac{n}{|n|}}$$

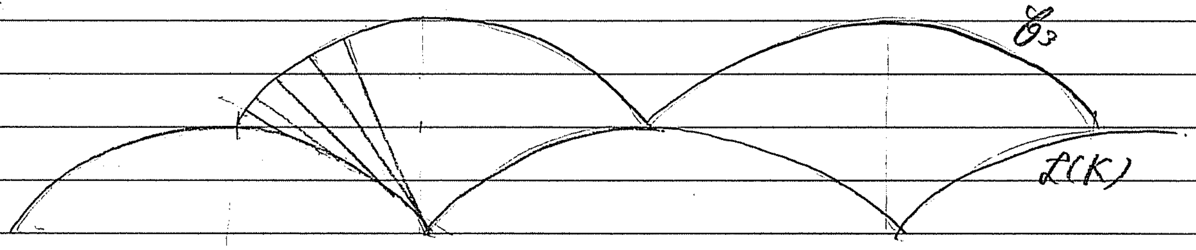
$$= \begin{pmatrix}t-\sin t\\ 1-\cos t\end{pmatrix} + 2\begin{pmatrix}\sin t\\ \cos t-1\end{pmatrix} = \begin{pmatrix}t+\sin t\\ \cos t-1\end{pmatrix}$$

よって $\xi: \begin{cases}\xi = t+\sin t\\ \eta = \cos t - 1\end{cases}$ になって

$$\begin{cases} \xi = \{(\theta+\pi) - \sin(\theta+\pi)\} - \pi \\ \eta = +1 - \cos(\theta+\pi) - 2 \end{cases}$$

$$\Leftrightarrow \begin{cases} \xi + \pi = (\theta+\pi) - \sin(\theta+\pi) \\ \eta + 2 = 1 - \cos(\theta+\pi) \end{cases}$$

$\mathscr{C}_3$ は $\begin{cases} x \text{方向} & -\pi \\ y \text{方向} & -2 \end{cases}$ 平行移動



Task 1.2

$\mathscr{C} : r = f(\theta)$

(1) 曲率半径 $\rho = \dfrac{(r^2 + r'^2)^{\frac{3}{2}}}{|r^2 + 2r'^2 - rr''|}$ show!

$\mathscr{C}$ を p-表示して $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$\dot{x} = \dot{r}\cos\theta - r\sin\theta$, $\dot{y} = \dot{r}\sin\theta + r\cos\theta$

$\ddot{x} = \ddot{r}\cos\theta - \dot{r}\sin\theta - \dot{r}\sin\theta - r\cos\theta = \ddot{r}\cos\theta - 2\dot{r}\sin\theta - r\cos\theta$

$\ddot{y} = \ddot{r}\sin\theta + \dot{r}\cos\theta + \dot{r}\cos\theta - r\sin\theta = \ddot{r}\sin\theta + 2\dot{r}\cos\theta - r\sin\theta$

曲率 K について

$\begin{vmatrix} \dot{x} & \ddot{x} \\ \dot{y} & \ddot{y} \end{vmatrix} = \dot{x}\ddot{y} - \ddot{x}\dot{y}$

$\ddot{r}\dot{r}$ $\cos\theta\sin\theta - \cos\theta\sin\theta = 0$

$\ddot{r}r$ $-\sin^2\theta - \cos^2\theta = -1$

$\dot{r}^2$ $2\cos^2\theta + 2\sin^2\theta = 2$

$\dot{r}r$ $-3\sin\theta\cos\theta + 3\sin\theta\cos\theta = 0$

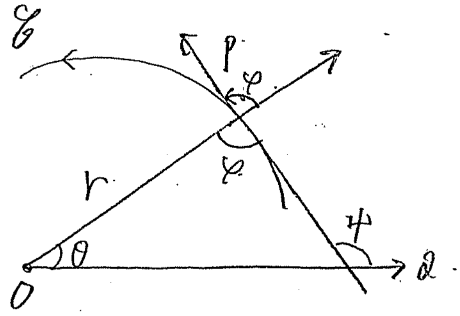
r^2 $\sin^2\theta + \cos^2\theta = 1$

$\therefore \dot{x}\ddot{y} - \ddot{x}\dot{y} = -\ddot{r}r + 2\dot{r}^2 + r^2$

また $\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2$

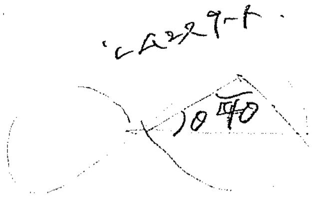
$\therefore K = \dfrac{r^2 + 2\dot{r}^2 - r\ddot{r}}{(\dot{r}^2 + r^2)^{\frac{3}{2}}}$ より 成立.

(2)



$\theta+\varphi=\psi$ より $\varphi=\psi-\theta$

$$\tan\varphi=\tan(\psi-\theta)=\frac{\tan\psi-\tan\theta}{1+\tan\psi\tan\theta}=\frac{\frac{y'}{x'}-\tan\theta}{1+\frac{y'}{x'}\tan\theta}$$



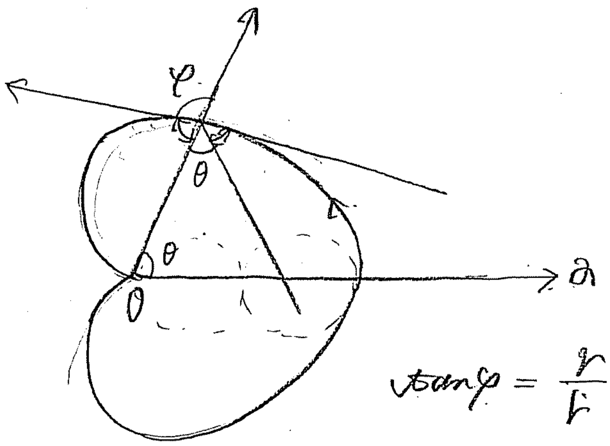
ここで $\dfrac{y'}{x'}=\dfrac{r'\sin\theta+r\cos\theta}{r'\cos\theta-r\sin\theta}$

$\cos\theta\neq0$ の下で $\dfrac{y'}{x'}=\dfrac{r'\tan\theta+r}{r'-r\tan\theta}$

$$\therefore\ \tan\varphi=\frac{\frac{r'\tan\theta+r}{r'-r\tan\theta}-\tan\theta}{1+\frac{r'\tan\theta+r}{r'-r\tan\theta}\tan\theta}=\frac{r+r\tan^2\theta}{r'-r\tan\theta+r'\tan^2\theta+r\tan\theta}$$

$$=\frac{r(1+\tan^2\theta)}{r'(1+\tan^2\theta)}=\frac{r}{r'}$$

(3)



$\mathcal{C}$: Cardioid

$r=1+\cos\theta$

接線と動径のなす角をφとして

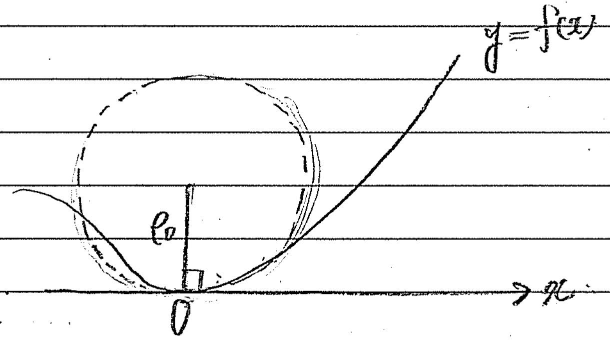
$$\tan\varphi=\frac{r}{r'}=\frac{1+\cos\theta}{\sin\theta}=-\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}=-\frac{1}{\tan\frac{\theta}{2}}$$

入・反射角はいずれも$\pi-\varphi$

よって光線のなす角は

$$\pi-2(\pi-\varphi)=2\varphi-\pi=2\left(\frac{\pi}{2}+\frac{\theta}{2}\right)-\pi=\theta$$

Task 1.3



$$\rho_0 = \lim_{x\to 0}\left|\frac{x^2}{2y}\right|$$

fが2階 contin. dif'ble だから
f, f', f'' は contin.

$y=f(x)$の曲率 $K = \dfrac{\ddot{y}}{(1+\dot{y}^2)^{\frac{3}{2}}} = \dfrac{f''(x)}{(1+f'(x)^2)^{\frac{3}{2}}}$

よって $\rho = \dfrac{|(1+f'(x)^2)^{\frac{3}{2}}|}{f''(x)}$

$f'(0)=0$ ∴ $\rho_{x=0} = \dfrac{1}{|f''(0)|}$

2次の接触
{接線, 曲率円} を共有
一般に n次の接触が可能

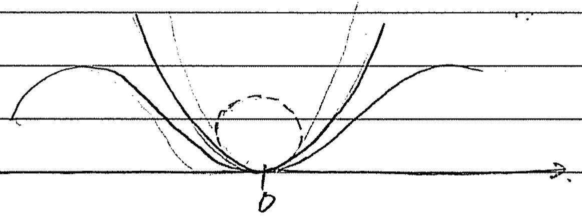
$f(x)$についての仮定から

$y = f(x) = f(0) + \dfrac{f'(0)}{1!}x + \dfrac{f''(\theta x)}{2!}x^2 \quad \theta\in(0,1)$

$= \dfrac{f''(\theta x)}{2!}x^2$ ∴ $f''(\theta x) = \dfrac{2y}{x^2}$

f''は contin. だから $f''(0) = \lim\limits_{x\to 0} f''(\theta x) = \lim\limits_{x\to 0}\dfrac{2y}{x^2}$

∴ $\rho_{x=0} = \dfrac{1}{|f''(0)|} = \lim\limits_{x\to 0}\left|\dfrac{x^2}{2y}\right|$.



$1+kx^2 \le \cos x \iff 1-\cos x \le -kx^2$
$-k = l$ と書けば lの範囲を求めればよい.
$l>0$が必要
$y = 1-\cos x = f(x)$, $y = lx^2 = g(x)$ とすると
f, g は仮定をみたすから それぞれの

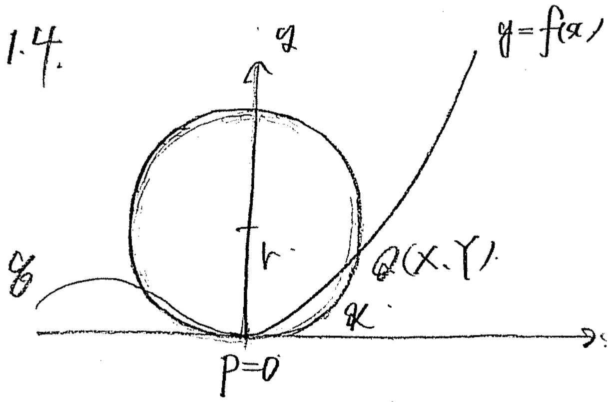
曲率半径を ρ^g, ρ^f として

$$\rho^f_{x=0} = \lim_{x\to 0}\left|\frac{x^2}{2y}\right| = \lim \frac{x^2}{2(1-\cos x)} = 1$$

また $\rho^f_{x=0} = \lim_{x=0}\left|\frac{x^2}{2y}\right| = \lim\left|\frac{x^2}{2lx^2}\right| = \frac{1}{2l}$

$\rho^g_{x=0} \le \rho^f_{x=0}$ より $\frac{1}{2l} \le 1$ ∴ $l \ge \frac{1}{2}$

Task 1.4.



$\mathcal{C}$, $\mathcal{K}$ の接点Pを原点Oにし、
接線をx軸に重ねる。
$\mathcal{C}$ が原点の近くで $y=f(x)$ と表されるとき
$K = \frac{\ddot{y}}{(1+\dot{y}^2)^{\frac{3}{2}}}$ より $\rho = \frac{(\quad)^{\frac{3}{2}}}{\ddot{y}}$

$x=0$ とすれば $\rho_{x=0} = \frac{1}{|\ddot{y}|}$

又 $\mathcal{K}$ の中心はy軸上にあるから、半径をrとして、中心(0, r)

∴ $\mathcal{K}$: $x^2+(y-r)^2=r^2$

$\mathcal{C}$, $\mathcal{K}$ の交点を $Q(X, Y)$ とすれば

$Y=f(X)$, $X^2+(Y-r)^2=r^2$

第2式より

$$r = \frac{X^2+Y^2}{2Y}$$

$Q \to P$ のとき $X \to 0$ になるから

$r = \frac{X^2+Y^2}{2Y}$ の $X \to 0$ のときの極限を考える

$X \to 0$ のとき $Y \to 0$, $\dot{Y} \to 0$

$$\lim \frac{X^2+Y^2}{2Y} = \lim \frac{2X+2Y\dot{Y}}{2\dot{Y}} = \lim \frac{X+Y\dot{Y}}{\dot{Y}} = \lim \frac{1+\dot{Y}^2+Y\ddot{Y}}{\ddot{Y}}$$

$$= \frac{1}{\ddot{Y}_{X=0}}$$

よって $r \xrightarrow[X\to 0]{} \frac{1}{\ddot{Y}_{X=0}} = \frac{1}{f''(0)} = \rho_{x=0}$

$\mathscr{C}: y=\cos kx$

$\mathscr{C}_1$ として $y=1-\cos kx$ を考えると、仮定をみたす。

$\dot{y}=k\sin kx$, $\ddot{y}=k^2\cos kx$

$\therefore \ddot{y}_{x=0}=k^2$ 2点の近づけ方は任意。

$\therefore \rho_{x=0}=\dfrac{1}{k^2}$

Task 1.5

$\mathcal{E}: \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

$x=a\cos t$. $y=b\sin t$

$\dot{x}=-a\sin t$. $\dot{y}=b\cos t$

$\ddot{x}=-a\cos t$ $\ddot{y}=-b\sin t$

$K=\dfrac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}$ について、 $\dot{x}^2+\dot{y}^2=a^2\sin^2 t+b^2\cos^2 t$

$\dot{x}\ddot{y}-\ddot{x}\dot{y}=ab\sin^2 t+ab\cos^2 t=ab$

$\therefore K=\dfrac{ab}{(a^2\sin^2 t+b^2\cos^2 t)^{\frac{3}{2}}}$

分母、分子に $a^3b^3=(a^2b^2)^{\frac{3}{2}}$ をかけて

$K=\dfrac{a^4b^4}{(a^4b^2\sin^2 t+a^2b^4\cos^2 t)^{\frac{3}{2}}}=\dfrac{a^4b^4}{(a^4y^2+b^4x^2)^{\frac{3}{2}}}$

$\therefore \rho=\dfrac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4}$

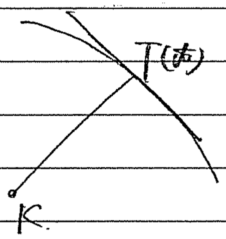
曲率中心

接vector $t=\begin{pmatrix}-a\sin t\\ b\cos t\end{pmatrix}$ 法線vector $n=\begin{pmatrix}-b\cos t\\ -a\sin t\end{pmatrix}$

$|n|^2=a^2\sin^2 t+b^2\cos^2 t$

n を正規化して

$u_n=\dfrac{1}{(a^2\sin^2 t+b^2\cos^2 t)^{\frac{1}{2}}}\begin{pmatrix}-b\cos t\\ a\sin t\end{pmatrix}$



$\overrightarrow{TK}=\rho u_n$

$=\dfrac{a^2\sin^2 t+b^2\cos^2 t}{a^2}\begin{pmatrix}-b\cos t\\ -a\sin t\end{pmatrix}$

$\therefore K(\xi,\eta)$ とすれば

$\overrightarrow{OK} = \overrightarrow{OT} + \overrightarrow{TK}$ より

$$\xi = a\cos\varphi + \left(-\frac{a^2\sin^2\varphi + b^2\cos^2\varphi}{ab}\right)\cdot b\cos\varphi$$

$$= \frac{1}{ab}\left(a^2 b\cos\varphi - a^2 b\sin^2\varphi\cos\varphi - b^3\cos^3\varphi\right)$$

$$= \frac{1}{a}\left(a^2\cos^3\varphi - b^2\cos^3\varphi\right) = \frac{a^2-b^2}{a}\cos^3\varphi$$

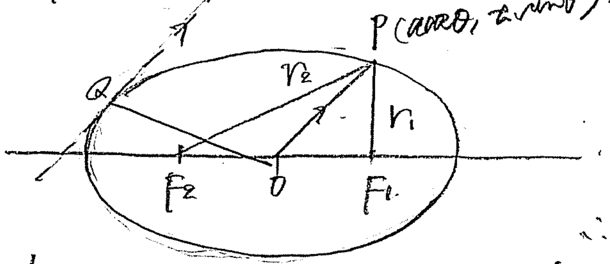
$$\eta = \cdots = \frac{a^2-b^2}{b}\sin^3\varphi$$

$\cos^3\varphi = \dfrac{a}{a^2-b^2}\xi$ より $\cos^2\varphi = \left(\dfrac{a}{a^2-b^2}\xi\right)^{\frac{2}{3}}$

$$\sin^2\varphi = \left(\frac{b}{a^2-b^2}\eta\right)^{\frac{2}{3}}$$

$$\therefore \left(\frac{a\xi}{a^2-b^2}\right)^{\frac{2}{3}} + \left(\frac{b\eta}{a^2-b^2}\right)^{\frac{2}{3}} = 1$$

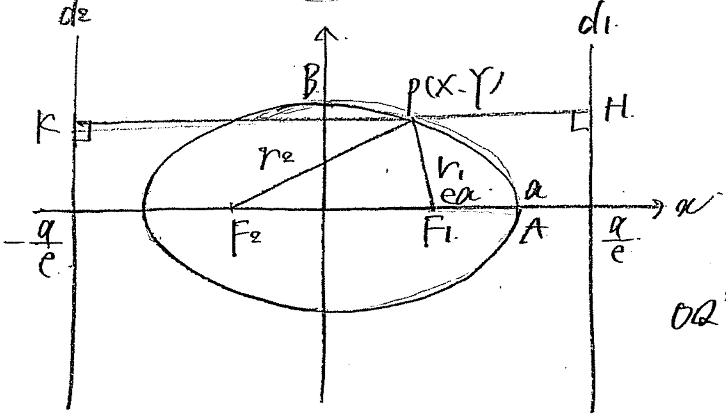
共役半径定理



OQ を OP の共役半径

$\overrightarrow{OP} = \begin{pmatrix} a\cos\theta \\ b\sin\theta \end{pmatrix}$ ならば $\overrightarrow{OQ} = \begin{pmatrix} a\cos(\theta+\frac{\pi}{2}) \\ b\sin(\theta+\frac{\pi}{2}) \end{pmatrix}$

$\therefore OQ^2 = a^2\sin^2\theta + b^2\cos^2\theta$

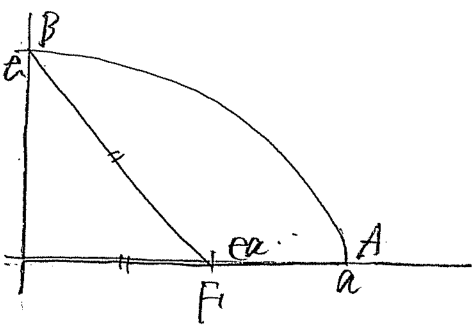


また $r_1 = PF_1 = PH\cdot e = a - ex$

$r_2 = PF_2 = PK\cdot e = a + ex$

$\therefore r_1 r_2 = a^2 - e^2x^2 = a^2(1 - e^2\cos^2\theta)$

$$OQ^2 = a^2\sin^2\theta + b^2\cos^2\theta$$
$$= a^2\left(\sin^2\theta + \frac{b^2}{a^2}\cos^2\theta\right)$$
$$= a^2\left(1 - \cos^2\theta + \frac{b^2}{a^2}\cos^2\theta\right)$$
$$= a^2\left\{1 - \cos^2\theta\,\frac{a^2-b^2}{a^2}\right\}$$



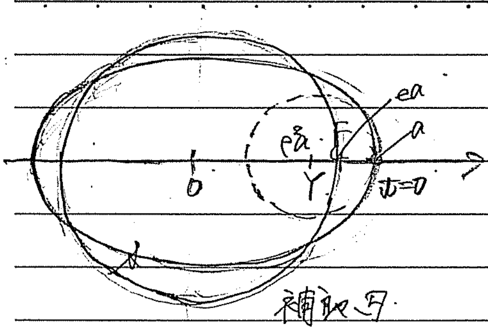
$e^2a^2 + b^2 = a^2$ より

$$\frac{a^2-b^2}{a^2} = e^2$$

$\therefore OQ^2 = a^2(1 - e^2\cos^2\theta)$

$OQ^2 = r_1 r_2$

$$\rho = \frac{OQ^3}{ab} = \frac{(r_1 r_2)^{\frac{3}{2}}}{ab}$$



$$\xi = \frac{a^2-b^2}{a}\cos^2\varpi$$

$\varpi=0$のとき $\xi = \frac{a^2-b^2}{a} = \frac{a^2e^2}{a} = ae^2$

補助円 $OY\cdot OA = OF^2$ より Yは Aの○に関する反転

Lecture 2 Area

Task 2.1 (1) $r = 3+\cos\theta$

面積要素 $dS = \frac{1}{2}r^2 d\theta$

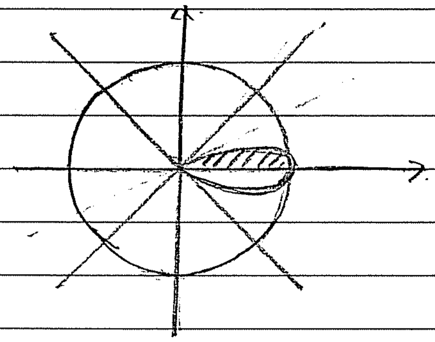
$$S = 2\int_{\theta=0}^{\theta=\pi} dS = \int_0^{\pi}(3+\cos\theta)^2 d\theta = \frac{19}{2}\pi$$

(2) $r = \cos 4\theta = g(\theta)$　　rは実数直線上のXより

$$g\left(\frac{\pi}{2}-\theta\right) = \cos 4\left(\frac{\pi}{2}-\theta\right) = \cos(2\pi-4\theta) = \cos 4\theta = g(\theta)$$

$g(-\theta) = g(\theta)$ より

$\theta = \frac{\pi}{4}$ について対称，始線についても対称

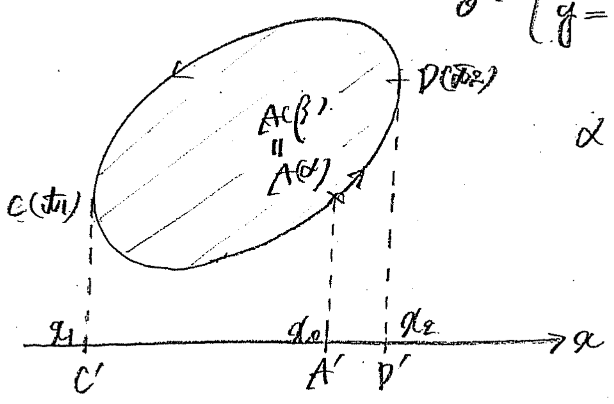


$$dS = \frac{1}{2}r^2 d\theta = \frac{1}{2}\cos^2 4\theta\, d\theta$$

$$S = 16\int_0^{\frac{\pi}{8}}\frac{1}{2}\cos^2 4\theta\, d\theta = 8\int_0^{\frac{\pi}{8}}\cos^2 4\theta\, d\theta = \frac{\pi}{2}$$

Fundamental Formula of Area

$$\mathscr{C}: \begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

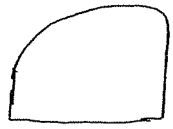


$$\alpha < t_2 < t_1 < \beta$$

$$S_1 = S_{AA'D'D} = \int_{x_0}^{x_2} y\,dx = \int_{\alpha}^{t_2} y \frac{dx}{dt}\,dt$$

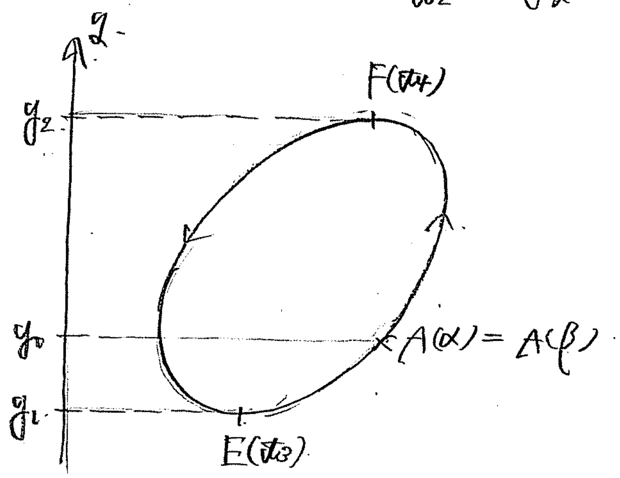
$$S_2 = S_{CC'D'D} = \int_{x_1}^{x_2} y\,dx = -\int_{x_2}^{x_1} y\,dx$$

$$= -\int_{t_2}^{t_1} y \frac{dx}{dt}\,dt$$



$$S_3 = S_{CC'A'A} = \int_{x_1}^{x_0} y\,dx = \int_{t_1}^{\beta} y \frac{dx}{dt}\,dt$$

$$S = S_2 - S_1 - S_3 = -\int_{t_2}^{t_1} - \int_{\alpha}^{t_2} - \int_{t_1}^{\beta} = -\int_{\alpha}^{\beta} y \frac{dx}{dt}\,dt$$



$$S = \int_{y_0}^{y_2} + \int_{y_1}^{y_0} - \int_{y_1}^{y_2}$$

$$= \int_{\alpha}^{t_4} + \int_{t_3}^{\beta} - \left(-\int_{t_4}^{t_3} \right)$$

$$= \int_{\alpha}^{t_4} + \int_{t_3}^{\beta} + \int_{t_4}^{t_3} = \int_{\alpha}^{\beta} x \frac{dy}{dt}\,dt$$

$$S = -\int_{\alpha}^{\beta} y \frac{dx}{dt}\,dt = \int_{\alpha}^{\beta} x \frac{dy}{dt}\,dt$$

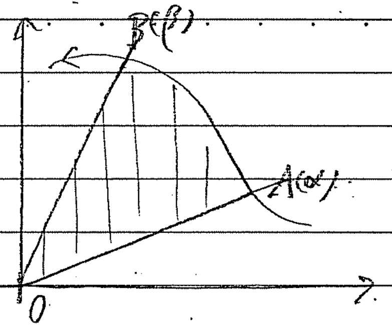
また、 $S = \frac{1}{2}\left\{ \int_{\alpha}^{\beta} x\dot{y}\,dt - \int_{\alpha}^{\beta} y\dot{x}\,dt \right\} = \frac{1}{2}\int_{\alpha}^{\beta} (x\dot{y} - y\dot{x})\,dt$

$$= \frac{1}{2}\int_{\alpha}^{\beta} (x\,dy - y\,dx)$$

また、 $x\dot{y} - y\dot{x}$ について

$$\frac{d}{dt}\frac{y}{x} = \frac{\dot{y}x - y\dot{x}}{x^2} \quad \text{より} \quad x\dot{y} - \dot{x}y = x^2\left(\frac{d}{dt}\frac{y}{x}\right)$$

$$\therefore S = \frac{1}{2}\int_{\alpha}^{\beta} x^2 \left(\frac{d}{dt}\frac{y}{x}\right) dt$$



$\theta : \alpha-1 \to \alpha$ OA

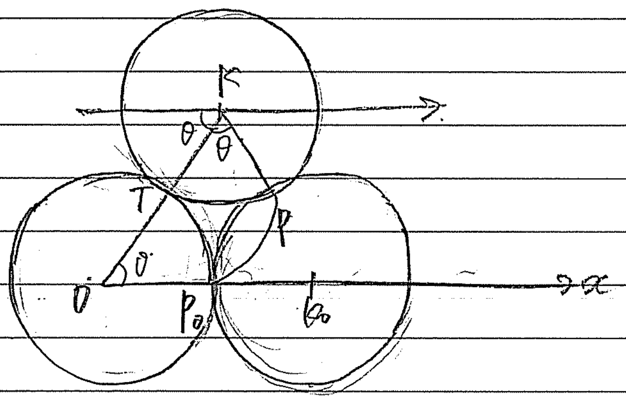
$\theta : \alpha \to \beta$ $\overset{\frown}{AB}$

$\theta : \beta \to \beta+1$ OB

$$\int_{\alpha-1}^{\beta+1} = \int_{\alpha-1}^{\alpha} + \int_{\alpha}^{\beta} + \int_{\beta}^{\beta+1}$$

$$S = \frac{1}{2}\int_{\alpha}^{\beta} x^2 \left(\frac{d}{d\theta}\frac{y}{x}\right) d\theta$$

Task



$\arg \overrightarrow{KP} = \pi + 2\theta$ より

$$\overrightarrow{KP} = \begin{pmatrix} \cos(\pi+2\theta) \\ \sin(\pi+2\theta) \end{pmatrix} = \begin{pmatrix} -\cos 2\theta \\ -\sin 2\theta \end{pmatrix}$$

また $\overrightarrow{OK} = 2\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ より $\overrightarrow{OP} = 2\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + \begin{pmatrix} -\cos 2\theta \\ -\sin 2\theta \end{pmatrix}$

$$\begin{cases} x = 2\cos\theta - (2\cos^2\theta - 1) \\ y = 2\sin\theta - 2\sin\theta\cos\theta \end{cases}$$

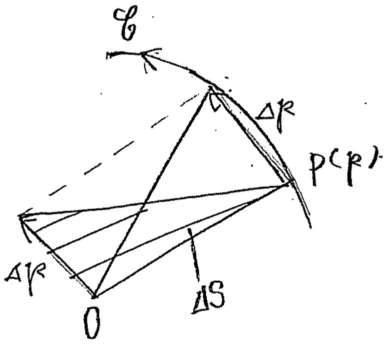
$x\,dy = (2\cos\theta - \cos 2\theta)(2\cos\theta - 2\cos 2\theta)\,d\theta = (4\cos^2\theta - 6\cos\theta\cos 2\theta + 2\cos^2 2\theta)\,d\theta$

$y\,dx = (2\sin\theta - \sin 2\theta)(-2\sin\theta + 2\sin 2\theta)\,d\theta = (-4\sin^2\theta + 6\sin\theta\sin 2\theta - 2\sin^2 2\theta)\,d\theta$

$\therefore x\,dy - y\,dx = \{6 - 6(\cos\theta\cos 2\theta - \sin\theta\sin 2\theta)\}\,d\theta = (6 - 6\cos 3\theta)\,d\theta$

$\therefore S = \frac{1}{2}\int_{\theta=0}^{\theta=2\pi}(x\,dy - y\,dx) = \frac{1}{2}\int_0^{2\pi} 6(1-\cos 3\theta)\,d\theta$

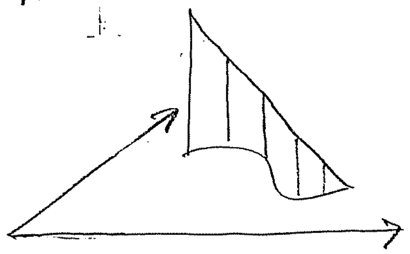
$= 3\left[\theta - \frac{1}{3}\sin 3\theta\right]_{\theta=0}^{\theta=2\pi} = 6\pi$

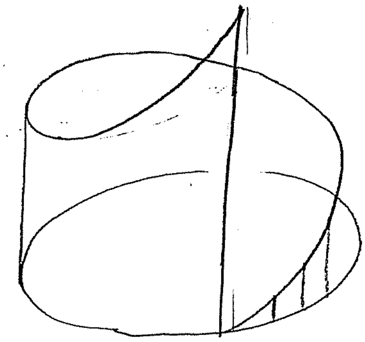


$$2\Delta S = \left[r \times \Delta r \right]_z$$

$$= \left[\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \times \begin{pmatrix} \Delta x \\ \Delta y \\ 0 \end{pmatrix} \right]_z = \begin{pmatrix} 0 \\ 0 \\ x\Delta y - y\Delta x \end{pmatrix}_z = x\Delta y - y\Delta x$$

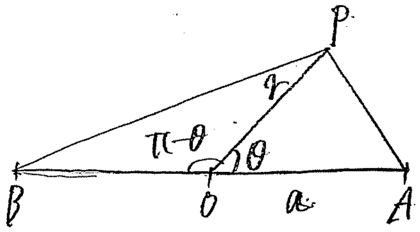
線積分のイメージ





閉曲線上に壁をつくる

Task 2.3　極座標、余弦定理　閉曲線大全



$PA \cdot PB = a^2$

$PA^2 = a^2 + r^2 - 2ar\cos\theta$

$PB^2 = a^2 + r^2 - 2ar\cos(\pi - \theta) = a^2 + r^2 + 2ar\cos\theta$

$\therefore PB^2 + PA^2 = (a^2+r^2)^2 - 4a^2r^2\cos^2\theta = a^4$

$r^4 + 2a^2r^2 - 4a^2r^2\cos^2\theta = 0$

$r \neq 0$ のとき $r^2 = 4a^2\cos^2\theta - 2a^2 = 2a^2\cos 2\theta$

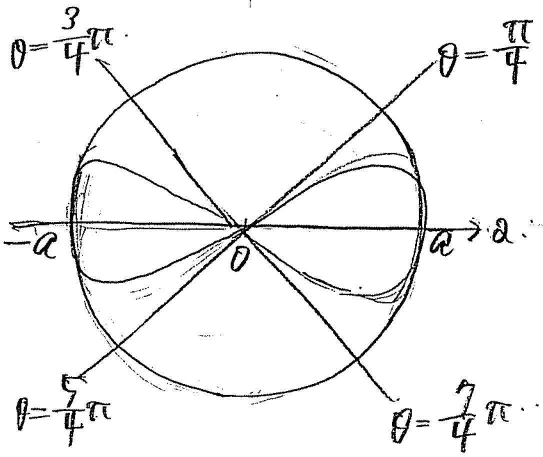
$r = 0$ のとき $P = O$

$r = a\sqrt{2\cos 2\theta}$

$-\pi \leq \theta \leq \pi \iff -2\pi < 2\theta \leq 2\pi$

$\cos 2\theta \geq 0$ だから

$$\begin{cases} -2\pi < 2\theta \leq -\frac{3}{2}\pi \\ -\frac{3}{2}\pi < 2\theta \leq \frac{\pi}{2} \\ \frac{1}{2}\pi < 2\theta \leq 2\pi \end{cases} \iff \begin{cases} -\pi < \theta \leq -\frac{3}{4}\pi \\ -\frac{3}{4}\pi < \theta \leq \frac{\pi}{4} \\ \frac{\pi}{4} < \theta \leq \pi \end{cases}$$



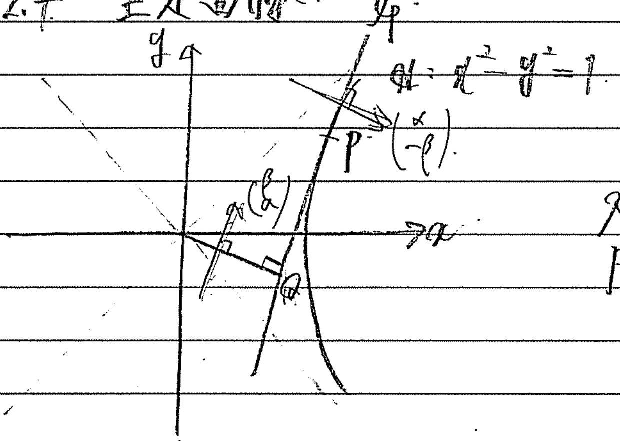
$a = 1$ として $r = \sqrt{2\cos 2\theta}$

$dS = \frac{1}{2} r^2 d\theta$ より

$S = 4\int_0^{\frac{\pi}{4}} dS$

$= 2a^2$

Task 2.4 垂足曲線



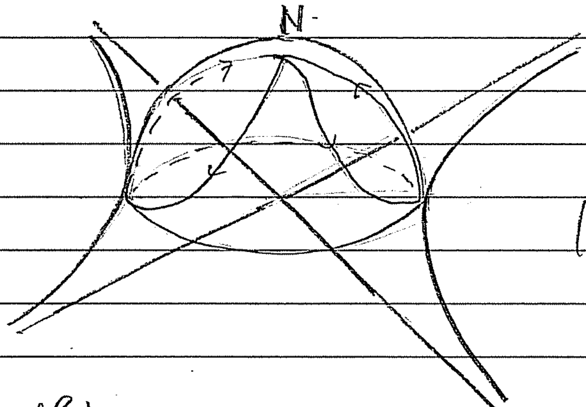
点Qのx軸対称点は
Hにおける点Pの反転

$H: x^2-y^2=1$

$P(\alpha, \beta)$、$Q(X, Y)$

$\ell_P : \alpha x-\beta y=1$

漸近線は無限遠点における接線
原点へで接点し



lemniscate
リボン

OQは $\begin{pmatrix}\beta\\\alpha\end{pmatrix}$ を法線vectorにもつから. $\beta x+\alpha y=0$

よって. $X\alpha-Y\beta=1$

$X\beta+Y\alpha=0 \Leftrightarrow Y\alpha+X\beta=0$

$$\alpha=\frac{\begin{vmatrix}1 & -Y\\0 & X\end{vmatrix}}{\begin{vmatrix}X & -Y\\Y & X\end{vmatrix}}=\frac{X}{X^2+Y^2},\quad \beta=\frac{-Y}{X^2+Y^2}\qquad X^2-Y^2=(X^2+Y^2)^2$$

$P(\alpha, \beta)$は $\alpha^2-\beta^2=1$ で成立し

$$\frac{X^2}{(X^2+Y^2)^2}-\frac{Y^2}{(X^2+Y^2)^2}=1.$$