cover

Mathematical Structure, Structural Math.

YAMASHITA, Koichiro

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2012/03/11(Sun) at Shinjuku

Outline

Contents

Uram's Dilemma

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... In mathematics one becomes married to one's own little field. Because of this, the judgement of value in mathematical research is becoming more and more difficult, and most of us are becoming mainly technicians.

Davis *et al*: The Mathematical Experience.

How many mathematics are there?

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  $60 \times 2 \times 42 = ?$ (`classifications2000.pdf`)

But ...

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ディヴィッド ドイッチュ「世界の究極理論は存在するか」

かなりノーテン ◯ ではあるが...

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- . for each $\mathscr{C}$-object A, an arrow $A \xrightarrow{\mathrm{id}_A} A$, called the $\mathscr{C}$-**identity** on A.
- a composition law associating with each arrow $A \xrightarrow{f} B$ and each arrow $B \xrightarrow{g} C$ an arrow $A \xrightarrow{g \circ f} C$, called the composite of f and g.

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$$h \circ (g \circ f) = (h \circ g) \circ f$$

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Let's visualize it!

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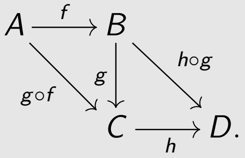

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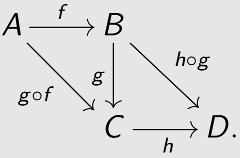

Identity

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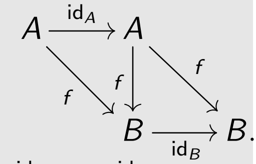

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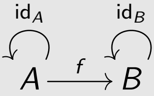

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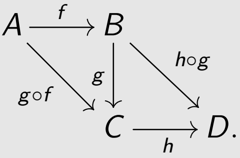

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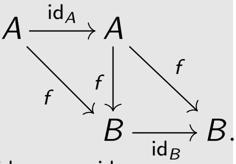

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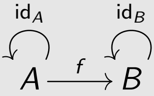

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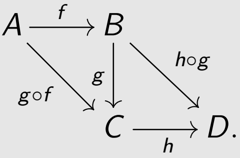

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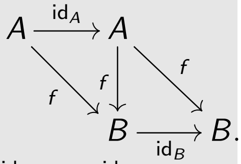

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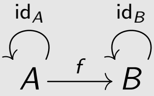

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Fin is defined by replacing "all set" in **Set** by all finite sets as object.

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- $\mathbb{R}^+$ is the set of all positive real numbers. $(\mathbb{R}^+, \times, 1)$ is a group. Also infinite and commutative.
- Let p a prime, and the set $E_p^\times \stackrel{\text{def}}{=} \{1, 2, \ldots, p-1\}$. Then $(E_p, \times_p, 1)$ is a group ($\times_p$ means production mod p). This group is commutative, but finite.
- $SL_2(\mathbb{R})$ is the set of 2-dimensional square matrices with real components, whose determinant are unity. $(SL_2(\mathbb{R}), \cdot, \mathbf{I})$ is a infinite group, which is non-commutative.

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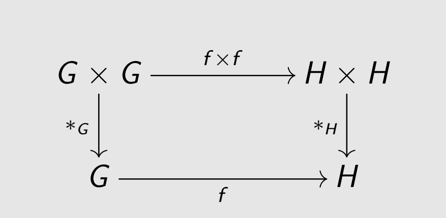

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Let's look at some categories without imperialism.

Contents

Order

Categories are ... very ... familiar !

Definition (preorder)

Let R is a binary relation on a set S(*i.e.* subset of $S \times S = S^2$). We say R is reflexive if for any x, xRx, and is transitive if xRy and yRz implies xRz for any $x, y, z \in S$. A set S with a reflexive, transitive relation R on it is a structure $(S,\ R)$ called a preordered set.

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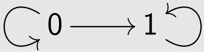
Example

$S = \{0, 1\}$ and $0R0$, $0R1$, $1R1$ on S.



Definition (poset, toset)

Of binary relation R, if xRy and yRx implies $x = y$, R is antisymmetric relation.
A preordered set $S = (S, R)$ for which R is antisymmetric is called a partially ordered set or a poset.
If, for any elements x and y of poset S, one and only one of xRy, $x = y$, yRx holds, S is called a totally ordered set or toset.

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Example

For any set S, the set of all subsets of S is called power set of S, written as $\mathscr{P}(S)$. The pair $(\mathscr{P}(S), \subseteq)$ is a poset.
If $\leq$ is the usual ordering, the pair $(\mathbb{R}, \leq)$ is a toset.

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It is very casual, isn't it? □

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$S = \{a, b, c\}$ has $2^3 = 8$ subsets, so $\sharp\mathscr{P}(S) = 8$. Diagram below is the poset category $(\mathscr{P}(S), \subseteq)$ (braces $\{\ \}$ are omitted).

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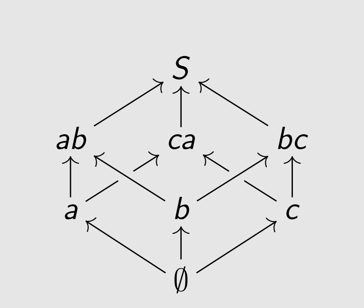

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Definition (alphabets, words, concatenation)

Let $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \ldots\}$ a set called alphabet-set, and strings (or lists) of elements of Σ called words on Σ. A word is expressed like $w = \mathtt{abbabc}$.

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And we allow empty word which contains no alphabet, written as ε.

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Every group is a monoid, but the inverse is not true.

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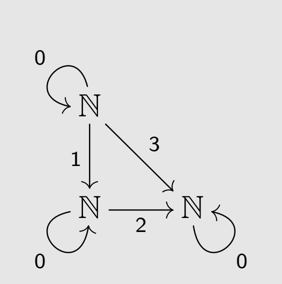

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Corollary

Let an alphabet-set $\Sigma = \{\mathtt{a}, \mathtt{e}, \mathtt{h}, \mathtt{k}, \mathtt{n}, \mathtt{o}, \mathtt{r}, \mathtt{t}, \mathtt{u}, \mathtt{v}, \mathtt{y}\}$*, then ...*
the sentence-word which I wish deeply to say everybody here belongs to the monoid-category $\mathscr{C}_{\Sigma^*}$*.*

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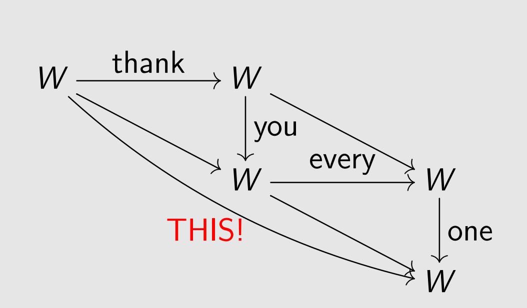

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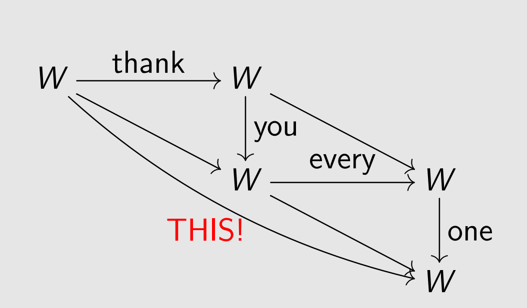

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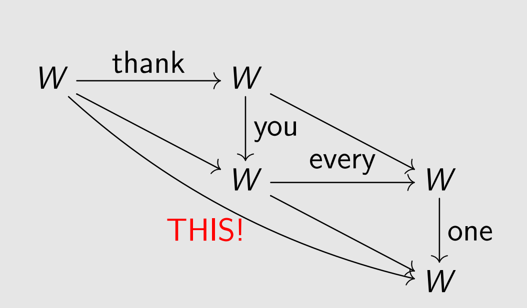

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Bye bye, everyone.